N-point amplitudes for d=2 c=1 Discrete States from String Field Theory.

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Abstract

Starting from string field theory for 2d gravity coupled to c=1 matter we analyze N-point off-shell tree amplitudes of discrete states. The amplitudes exhibit the pole structure and we use the oscillator representation to extract the residues. The residues are generated by a simple effective action. We show that the effective action can be directly deduced from a string field action in a special transversal-like gauge.

1   Introduction

In this paper we continue the study of the scattering amplitudes of discrete states in 2d string that was started in our previous paper [1], which in the following will be refereed as **I**. The characteristic property of the model is an appearance of discrete states which together with tachyon states exhaust the spectrum of physical states [2, 3, 4, 5]. Our starting point in **I** is the Witten-type String Field Theory [6] for this model [5]. The main reason to employ String Field Theory for this model is a necessity to have a definition for off-shell scattering amplitudes since the on-shell amplitudes are ill-defined [7]. The divergences of the amplitudes are caused by the resonance denominator phenomenon well-known in the usual field theories and occurred in the case of particle decays. A simple example is the Lee model [8]. Indeed, two-vectors of momentum-energy of discrete states

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form a two-dimensional lattice and summing up a pair of discrete momenta (according to the conservation law with the background charge) one gets as a rule the point of this lattice again. In other words, the internal lines of the graphs for tree-level amplitudes describe a propagation of real (non-virtual) "particles". Hence, calculating a scattering amplitude one unavoidably encounters the poles that leads to senseless results.

Starting with String Field Theory we have defined in **I** the 4-point scattering amplitudes off-shell and then have investigated their behavior near the mass-shell. In this paper we consider the N-point off-shell scattering amplitudes for discrete states and an effective action for the residues. In I we have used intensively the off-shell conformal method developed by Samuel et al. [9]. However, application of this method to the case of the N-point scattering amplitude meets serious analytical difficulties. By this reason here we will use the oscillator representation of 2d String Field Theory [10, 11]. It turns out that it is possible to separate the most singular part of N-point amplitudes. Moreover, the most singular part is determined by the structure constant of the discrete states operator algebra, which as it was shown in [12], is responsible for the hidden symmetry of the model. This gives a hint that the *most singular parts* of scattering amplitudes can be described by a simple effective theory. Note in this context that a few years ago Klebanov and Polyakov [12] proposed an effective action to describe the interaction of discrete states. To search for some sort of effective action we use a new gauge which we call a transversal-like gauge. We find that the most singular parts of tree-level discrete states off-shell amplitudes are generated by the part of the gauge-fixed action containing the discrete states fields only. We identify this piece of the gauge-fixed action with an effective action of the theory.

The paper is organized as follows. In Section 2 we investigate the singular behavior of an arbitrary N-point tree-level off-shell discrete states amplitude near mass-shell by using the oscillator representation of String Field Theory in the Siegel gauge. We find that the most singular part of the amplitude is the product of the poles in chanel variables times the product of structure constants, which enters in OPE of external states vertex operators. In Section 3 we investigate the 2d String Field Theory in the new gauge and separate the part responsible for the most singular parts of the graphs.

2 The N-point tree-level off-shell discrete states amplitudes in the Siegel gauge

In this section we shall describe the tree graphs of the model by using the oscillator representation of 2d String Field Theory.

The Feynman graphs we are going to examine are generated by the Witten-type action [6]

$$S = \int (\Phi * Q\Phi + \frac{2}{3}\Phi * \Phi * \Phi) \tag{1}$$

As usual, Φ is a string field and $\int$ and $*$ are well- known Witten integration and product operations. The action can be written in alternative form

$$S = {}_1\langle {}_2\langle I||\Phi\rangle_1 Q^{(2)}|\Phi\rangle_2 + \frac{2}{3} {}_1\langle {}_2\langle {}_3\langle V||\Phi\rangle_1|\Phi\rangle_2|\Phi\rangle_3, \tag{2}$$

where ${}_1\langle {}_2\langle I|$ is "identity" operator transforming kets into bras and ${}_1\langle {}_2\langle {}_3\langle V|$ is Witten three-vertex describing the interaction of string. The explicit form and properties for

$_1\langle_2\langle I|$ and $_1\langle_2\langle_3\langle V|$ for $D = 2$ had been discussed in [10, 11]. The propagator in Siegel gauge $b_0|\Phi\rangle = 0$ is equal to $b_0(L_0)^{-1}$.

In the modern setting, the discrete states appear as a non-trivial cohomology of BRST charge $Q : Q|\psi\rangle = 0$, $|\psi\rangle \neq Q|\lambda\rangle$, and they can be classified according to their ghost number. The nontrivial cohomology can be found for ghost number $0, ..., 3$ and it can be proved that not only the states with fixed momenta are in the spectrum but the whole spectrum is exhausted by these states plus the tachyon with non-discrete momentum. In the following we shall deal with compactified version of the model with matter coordinate x being compactified on $SU(2)$ radius. In this case all points of the spectrum can be described in unified fashion and belong to a single lattice. This physical states in the total ("matter" + ghosts) Fock space can be explicitly described in two ways: by using Shur polynomials [13] or in so called "material gauge" in terms of $SU(2)$ raising and lowering operators [12, 15, 16]. For our purposes the second description is more convenient.

The on-shell physical states are defined in terms of conformal fields Y [12, 15]

$$Y_{J,n}^{\pm}(z) = cW_{J,n}^{\pm}(z) = cV_{J,n}e^{\sqrt{2}(1\mp J)}. \tag{3}$$

The momenta of discrete states read:

$$(p, \varepsilon) = \sqrt{2}(n, \ 1 \mp J) \tag{4}$$

where

$$J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...; \quad n = -J, -J + 1, ..., J.$$

Hereafter we adopt the standard correspondence between conformal fields $Y_{J,n}^{\sigma}(z)$ ($\sigma = \pm$) and states $|Y_{J,n}^{\sigma}\rangle$ in the total Fock space:

$$\lim_{z \to 0} Y_{J,n}^{\sigma}(z)|0\rangle = |Y_{J,n}^{\sigma}\rangle, \tag{5}$$

where $|0\rangle$ is SL_2-invariant vacuum.

Hidden symmetry of the theory is encoded in structure constants of OPE for W-s:

$$W_{J_1,n_1}^{\sigma_1}(z)W_{J_2,n_2}^{\sigma_2}(0) = \frac{1}{z}\tilde{f}_{J_1n_1\sigma_1; \ J_2n_2\sigma_2}^{J_3n_3\sigma_3} W_{J_3,n_3}^{\sigma_3}(0) + ... \tag{6}$$

The explicit expressions for f-s had been obtained in [12]:

$$\tilde{f}_{J_1n_1+; \ J_2n_2+}^{J_3n_3+} = (J_2n_1 - J_1n_2)\delta_{J_1+J_2-1,J_3}\delta_{n_1+n_2,n_3},$$

$$\tilde{f}_{J_1n_1+; \ J_2n_2-}^{J_3n_3-} = (-J_1n_3 - J_3n_1)\delta_{J_1+J_3-1,J_2}\delta_{n_3-n_1,n_2} \tag{7}$$

and other f-s are equal to zero. It has been found in [18] that naive OPE of the vertex operators W does form well defined current algebra. To avoid this one need to multiply each of W by corresponding cocycle operator.

However the String Field Theory provides a rigid frame for string field and we cannot multiply by hand the vertex operators, which are on-shell string fields, by some cocycles. Then the expression for the structure constant in OPE of vertex operators are modified to be

$$f_{J_1n_1\sigma_1; \ J_2n_2\sigma_2}^{J_3n_3\sigma_3} = (-1)^{2J_1(J_2-n_2-1)} \tilde{f}_{J_1n_1\sigma_1; \ J_2n_2\sigma_2}^{J_3n_3\sigma_3}. \tag{8}$$
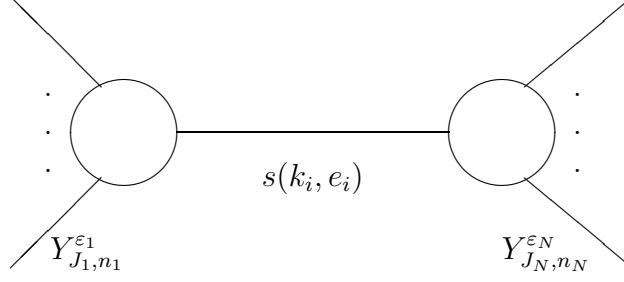
Figure 1: The definition of the variables $s(k_i, e_i)$

The structure constants (8) does not poses definite symmetry properties of indices, therefore they are not correspond to any algebra.

To set the convention we take any graph in the form:

$$A = {}_1\langle {}_2\langle {}_{i_1}\langle V| \ {}_{j_1}\langle {}_3\langle {}_{i_2}\langle V| \ \cdots \ {}_{j_{N-3}}\langle {}_{N-1}\langle {}_N\langle V|$$

$$\frac{b_0^{j_1}}{L_0^{j_1}}|I\rangle_{i_1}\rangle_{j_1} \ \frac{b_0^{j_2}}{L_0^{j_2}}|I\rangle_{i_2}\rangle_{j_2} \ \cdots \ \frac{b_0^{j_{N-3}}}{L_0^{j_{N-3}}}|I\rangle_{i_{N-3}}\rangle_{j_{N-3}} \ |Y_1\rangle_1 \ |Y_2\rangle_2 \ \cdots \ |Y_N\rangle_N \tag{9}$$

In the Liouville zero modes sector of Fock space we put the following normalization

$$\langle \varepsilon'|\varepsilon\rangle = \delta(\varepsilon' - \varepsilon) \tag{10}$$

$$p_\phi|\varepsilon\rangle = \varepsilon|\varepsilon\rangle \tag{11}$$

To describe the proper variables for off-shell tree amplitudes we specify the way of moving off the mass shell for discrete states. Although we are not going to use the methods of Conformal Field Theory we adopt the same definition for off-shell discrete states as in our previous paper [1] taking the discrete states in the form

$$Y_{J,n}^\pm = cV_{J,n}e^{\sqrt{2}(1\mp J)\phi} \tag{12}$$

we can relax the mass-shell condition by simply substituting: $\sqrt{2}(1\mp J)\phi \to \varepsilon\phi$ for some parameter ε. The fields

$$Y_{J,n,\sigma}^\varepsilon = cV_{J,n}e^{\varepsilon\phi}$$

will be associated with external legs of tree graphs. The subscript $\sigma = \pm$ indicates that ε lies in a small neighborhood of the point $\sqrt{2}(1 - \sigma J)$. It is obvious that as soon as the couple of external states $Y_{J,n,\sigma}^\varepsilon$ is given, the energy-momenta of any internal line is uniquely defined. In the following we shall use the compact notation for subscripts of discrete states $(J, n, \sigma) = a$. For an amplitude of a tree graph with N external off-shell states $Y_{a_r}^{\varepsilon_r}$, $r = 1, \ldots, N$ we use the notation $A_N(a_1, ..., a_N)$.

To analyze the pole structure of A_N it is convenient to use the variables (see Fig. 1)

$$s(k_i, e_i) = k_i^2 - (e_i - \sqrt{2})^2 + 2 \tag{13}$$

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associated with i-th internal line of the graph. Here k_i and e_i are the momentum and Liouville energy for the line. Short comment is in order. The unusual term in the definition of s_i has it origin in the linear term proportional to Q in the operator L_0:

$$L_0 = \frac{1}{2}[p_x^2 - (p_\phi - \sqrt{2})^2 + 2\hat{N}] \tag{14}$$

where

$$\hat{N} = \sum_{n=1}^{\infty}(\alpha_{-n}\alpha_n + \phi_{-n}\phi_n + nc_{-n}b_n + nb_{-n}c_n) \tag{15}$$

is the level number operator. It is easy to check that s_i defined by eq.(13) does not depend on what energy-momenta are taken to define it: from the "left" or from the "right" of the i-th line. When all the external states are on-shell then

$$s_i = s_i^0 = 2[(\sum n_r)^2 - (\sum(1 - \sigma_r J_r))^2 + 2\sum(1 - \sigma_r J_r)] \tag{16}$$

and with this notation the poles of a graph will appear as $1/(s_i - s_i^0) = 1/\delta s_i$. For each N-point tree graph the number of variables $s_i(k,e)$ is equal to $N - 3$ while the number of independent variables ε_r is $N - 1$ (N external legs minus one conservation low). In the following it will be convenient for us to parameterize each N-point amplitude by s_i. To have a one to one correspondence between $\{s_i\}$ and $\{\varepsilon_r\}$ we fix a pair of ε_r for two distinct vertices each having two external legs. By using the definition (13) we can obtain the relation between δs_i and $\delta\varepsilon_r = \varepsilon_r - \sqrt{2}(1 - \sigma_r J_r)$ up to terms of order $(\delta\varepsilon_r)^2$

$$s_i - s_i^0 = -2(e_i - \sqrt{2})\delta e_i + O((\delta e_i)^2),$$

$$e_i = \sum \varepsilon_r, \quad \delta e_i = \sum \delta\varepsilon_r, \tag{17}$$

where the sum in the second line of (17) is performed over all external lines standing from the left (or from the right) of i-th internal line.

In the following we shall examine only the graphs such that the corresponding amplitudes have poles in all the variables s_i which parameterize the graph. We shall call graphs of this type the *most singular graphs*. As it was shown in **I** and [7] there exist graphs which by dynamical reasons do not have poles in some s_i.

As a preliminary step let us consider the calculation of the typical element of a graph: a result of contraction of a pair of off-shell states with the three string Witten vertex ${}_1\langle{}_2\langle{}_3\langle V|$

$${}_1\langle{}_2\langle{}_{3'}\langle V|Y_{a_1}\rangle_1|Y_{a_2}\rangle_2|I\rangle_{3'}\rangle_3 \tag{18}$$

Here we have transformed the bras into kets by using the "identity " $|I\rangle\rangle$. As it was shown in [11] the "identity" operator and three-string vertex are BRST-invariant

$${}_1\langle{}_2\langle{}_3\langle V|(Q^{(1)} + Q^{(2)} + Q^{(3)}) = 0, \tag{19}$$

$$(Q^{(1)} + Q^{(2)})|I\rangle_1\rangle_2 = 0, \tag{20}$$

so for two on-shell states one has

$$Q^{(3)}\,{}_1\langle{}_2\langle{}_{3'}\langle V|Y_{a_1}\rangle_1|Y_{a_2}\rangle_2|I\rangle_{3'}\rangle_3 = 0. \tag{21}$$

Note, that energy conservation low in $|I\rangle\rangle$ and $\langle\langle\langle V|$ in presence of the background charge is modified to give: $\delta(\sum \varepsilon - 2\sqrt{2})$. The BRST-cohomology for ghost number 2 was described in [13] that results in:

$$_1\langle\,_2\langle\,_{3'}\langle V|Y_{a_1}\rangle_1|Y_{a_2}\rangle_2|I\rangle_{3'}\rangle_3 = f^b_{a_1 a_2} c_0 |Y_b\rangle_3 + Q|\Lambda_{a_1 a_2}\rangle_3. \tag{22}$$

Now we take into account the explicit dependence of the vertex $_1\langle\,_2\langle\,_{3'}\langle V|$ on the Liouville energy. Really, the relevant part of the vertex can be written as

$$_1\langle\,_2\langle\,_{3'}\langle V| \sim \int d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2\sqrt{2})$$

$$_1\langle\varepsilon_1|\,_2\langle\varepsilon_2|\,_3\langle\varepsilon_3| \exp[\sum_{r=1}^{3} N^{rr}_{00}(\varepsilon_r^2 - 2\sqrt{2}\varepsilon_r) + \sum_{r,s=1}^{3} \sum_{n=1}^{\infty} N^{rs}_{0n} \varepsilon_r \phi^{(s)}_n], \tag{23}$$

where N^{rr}_{00} and N^{rs}_{0n} are some numbers (the Neumann coefficients). We see that the vertex depends analytically on ε_r (at least when acting on finite states), hence it follows from eq.(22) and the definition of the off-shell states that

$$_1\langle\,_2\langle\,_{3'}\langle V|Y^{\varepsilon_1}_{J_1,n_1,\sigma_1}\rangle_1|Y^{\varepsilon_2}_{J_2,n_2,\sigma_2}\rangle_2|I\rangle_{3'}\rangle_3 - f_{123}|Y^{\varepsilon_1+\varepsilon_2}_{J_3,n_3,\sigma_3}\rangle_3 - Q|\Lambda^{\varepsilon_1+\varepsilon_2}_{J_1 n_1 \sigma_1, J_2 n_2 \sigma_2}\rangle_3 = O(\delta\varepsilon_1, \delta\varepsilon_2). \tag{24}$$

Next we note that the states Y^ε_a diagonalize L_0. For the state $|Y^e_a\rangle$ of given energy-momentum $(k, e) = (\sqrt{2}(n_1 + n_2), \varepsilon_1 + \varepsilon_2)$ we have according to definition (13) and (14): $L_0|Y^e_a\rangle = (\frac{1}{2}s(k,e) + \hat{N} - 1)|Y^e_a\rangle$ hence for a discrete states $Y^\pm_{J,n}$ one gets: $(\hat{N} - 1)|Y^\pm_{J,n}\rangle = -\frac{1}{2}s^0(n, J)|Y^\pm_{J,n}\rangle$. Taking into account our definition of the off-shell states we have

$$L_0|Y^e_a\rangle = \frac{1}{2}(s(k,e) - s^0)|Y^e_a\rangle = \frac{1}{2}\delta s(k,e)|Y^e_a\rangle. \tag{25}$$

Making use of eqs. (24) and (25) we see that the element (18) being multiplied by the propagator $\frac{b_0}{L_0}$ is equal to

$$_1\langle\,_2\langle\,_{3'}\langle V|Y^{\varepsilon_1}_{a_1}\rangle_1|Y^{\varepsilon_2}_{a_2}\rangle_2 \frac{b^{(3)}_0}{L^{(3)}_0}|I\rangle_{3'}\rangle_3 = \frac{2}{\delta s(k,e)} f^b_{a_1 a_2}|Y^e_b\rangle_3$$

$$+ \frac{b^{(3)}_0}{L^{(3)}_0} Q^{(3)}|\Lambda^e_{a_1 a_2}\rangle_3 + \frac{b^{(3)}_0}{L^{(3)}_0}[\delta\varepsilon_1|\Omega_{1a_1 a_2}\rangle + \delta\varepsilon_2|\Omega_{2a_1 a_2}\rangle] \tag{26}$$

with $k = \sqrt{2}(n_1 + n_2)$ and $e = \varepsilon_1 + \varepsilon_2$. The substitution of eq.(26) into the full graph leads to a representation of the original graph as a sum of new graphs. Let us extract only the terms in this sum which determine the most singular behavior of the original graph.

The first term in eq.(26) contains a simple pole in variable s. It diverges as soon as the external states 1 and 2 move on shell and therefore will give a contribution to the most singular part of the original graph.

The third term in eq.(26) does not lead to any infinities in on-shell limit: a possible pole originating from $\frac{1}{L_0}$ will be cancelled by factors $\delta\varepsilon_1$ and $\delta\varepsilon_2$. Therefore this term does not produce the most singular graphs.

The remaining second term in eq.(26) can be rewritten in the form

$$|\Lambda^e_{a_1 a_2}\rangle - Q\frac{1}{L_0}\rangle|\Lambda^e_{a_1 a_2}\rangle. \tag{27}$$

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The first term in eq.(27) is finite in on-shell limit. The second term contains the BRST operator Q which acts on one of the vertices of the remaining part of whole graph. Using the properties (20) and (19) one can pull Q through the graph until it acts on some external line. As a result we obtain the new series of graphs with cancelled propagator (owing to the identity $[Q, \frac{b_0}{L_0}] = 1$) as well as the series of graphs with operator Q acting on one of the external legs. Each of these new graphs will have the lower singular behavior as compared to the original graph.

So we see that the most singular behavior of original graph is determined by

$$A_N^{m.s.}(a_1, ..., a_N) = \frac{2}{s_{b_1} - s_{b_1}^0} f_{a_1 a_2}^{b_1} A_{N-1}(b_1, a_3, ..., a_N). \tag{28}$$

In the r.h.s. of this expression we have the product of the pole, the structure consistent and the $N-1$-point amplitude corresponding to the graph in which the element (18) is replaced by the discrete state $Y_{b_1}^{\varepsilon_1 + \varepsilon_2}$.

On the next step we can repeat the above procedure for A_{N-1}:

$$A_{N-1}^{m.s.}(b_1, a_3..., a_N) = \frac{2}{s_{b_2} - s_{b_2}^0} f_{b_1 a_3}^{b_2} A_{N-2}(b_2, a_4, ..., a_N). \tag{29}$$

Going step by step we come to the following expression for the most singular part of A_N:

$$A_N^{m.s.}(a_1, ..., a_N) = (\prod_{i=1}^{N-3} \frac{2}{s_{b_i} - s_{b_i}^0}) f_{a_1 a_2}^{b_1} f_{b_1 a_3}^{b_2} ... f_{b_{N-3} a_{n-1}}^{a_n} \tag{30}$$

3 Traversal-like gauge and effective Lagrangian

The aim of this section is to find a way to separate out the discrete states interaction singularities just on the Lagrangian level by choosing an appropriate gauge.

We put the following gauge condition on a string field $|\Phi\rangle = |\varphi\rangle + c_0 |\psi\rangle$:

$$\tilde{Q}|\varphi\rangle = 0 \tag{31}$$

$$|\psi\rangle - \text{arbitrary}, \tag{32}$$

where $\tilde{Q}$ is defined by the ghost zero modes expansion of Q : $Q = c_0 L_0 + b_0 M + \tilde{Q}$. Equation (31) is well known in the literature as defining the relative cohomology of Q [13]. However, we do not restrict the space of possible solutions for eq.(31) to be on-shell, i.e. we do not put the usual constraint: $L_0|\varphi\rangle = 0$, so $\tilde{Q}^2 = -L_0 M \neq 0$.

The gauge $\tilde{Q}|\varphi\rangle = 0$ is admissible. Indeed, the gauge transformation $\delta|\Phi\rangle = Q|\Lambda\rangle$ with $|\Lambda\rangle = |\lambda\rangle + c_0|\omega\rangle$ for the field $|\varphi\rangle$ gives

$$\delta|\varphi\rangle = \tilde{Q}|\lambda\rangle + M|\omega\rangle, \tag{33}$$

Suppose $\tilde{Q}|\varphi\rangle \neq 0$ then

$$\tilde{Q}(|\varphi\rangle + \delta|\varphi\rangle) = \tilde{Q}|\varphi\rangle - L_0 M|\lambda\rangle + \tilde{Q}M|\omega\rangle. \tag{34}$$

Put $|\omega\rangle = 0$, then the gauge condition will be satisfied by $|\lambda\rangle = M^{-1} L_0^{-1} \tilde{Q}|\varphi\rangle$. The crucial point here is if $M = \sum_n n c_{-n} c_n$ is invertible or not. The resolution actually depends on

the ghost number of a state $|\lambda\rangle$. In our case $N_{gh}(|\lambda\rangle) = 0$. We have not proved rigorously that M is invertible for this ghost number, but the analysis of a number of examples forces us to conjecture that this is actually the case. Hence, we conclude that our gauge condition eq.(31) can be fulfilled.

One important remark is now in order. It is obvious from the discussion in the previous section that the most singular are the graphs which join three discrete states in each three-string vertex. The explicit form of the structure constants [12] tells us that in this case the vertex has two $+$ legs and one $-$ leg and the propagator connect one $+$ and one $-$ state. If we denote by N_+ (N_-) the number of external $+$ ($-$) legs for a graph with V vertices, ν loops and $\mathcal{N} = \nu + V - 1$ internal lines, then we get: $V = N_- + \mathcal{N} = N_- + V - 1 + \nu$, i.e.,

$$N_- = 1 - \nu. \tag{35}$$

So, we recognize that just the tree graphs are of special significance in the model and in the following we shall convince ourselves by these graphs.

Thus, for our specific task we can ignore the ghost contribution and overcome the complete gauge fixing procedure restricting ourselves by the first step, i.e. by taking the partition function in the form:

$$Z = \int [\mathcal{D}\varphi][\mathcal{D}\psi]\delta(\tilde{Q}\varphi)e^{-S(\varphi,\psi)} \tag{36}$$

Now let us discuss the actual content of the gauge fixing condition $\tilde{Q}|\varphi\rangle = 0$. The string field in momentum representation can be written down in the form:

$$|\varphi\rangle = \int d\varepsilon \sum_{A,n} \varphi_{A,n}(\varepsilon) M_A(\alpha, \phi, b, c)|\sqrt{2}n, \varepsilon\rangle^{x,\phi}|1\rangle^{b,c}, \tag{37}$$

where $\varphi_{A,n}(\varepsilon)$ are the ordinary fields, M_A – some monomials in string (α_k, ϕ_k) and ghost normal modes, the ghost vacuum $|1\rangle^{bc}$ is defined via SL_2 invariant one by $|1\rangle^{bc} = c_1|0\rangle^{bc}$ and we have utilized the fact that the "space" is compactified on $SU(2)$ radius. The gauge fixing condition results in an infinite system of linear equations that can be written symbolically as

$$\varepsilon C_{AB}\varphi_{Bn}(\varepsilon) + D^n_{AB}\varphi_{Bn}(\varepsilon) = 0. \tag{38}$$

For each fixed values of n and ε and fixed mass level only a finite number of equations in (38) survives thus giving a possibility to solve the system. So, we see that not all the integration variables in $[\mathcal{D}\varphi]$ are independent but there is a arbitrariness in choosing the proper integration variables.

To exemplify our choice let us consider the mass level one. The $|\varphi\rangle$ component of the string field $|\Phi\rangle$ on this level is given by

$$|\varphi\rangle = \int d\varepsilon \sum_n [A_n(\varepsilon)\alpha_{-1} + D_n(\varepsilon)\phi_{-1}]|\sqrt{2}n, \varepsilon\rangle^{x,\phi}|1\rangle^{b,c} \tag{39}$$

The gauge condition results in single linear equation:

$$\sqrt{2}nA_n(\varepsilon) - (\varepsilon - 2\sqrt{2})D_n(\varepsilon) = 0. \tag{40}$$

For $n = 0$ the field $A_0(\varepsilon)$ remains to be an arbitrary function of ε while the value of ε for $D_0(\varepsilon)$ is fixed to be: $\varepsilon = 2\sqrt{2}$ that is nothing but the value defined by the mass-shell condition for this level:

$$2n^2 - \varepsilon(\varepsilon - 2\sqrt{2}) = 0. \tag{41}$$

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On the other hand, the state $\alpha_{-1}|0,\varepsilon\rangle^{x,\phi}|1\rangle^{b,c}$ corresponds to the off-shell discrete states $|Y_{1,0}^{\varepsilon}\rangle$, while

$$\phi_{-1}|0,2\sqrt{2}\rangle^{x,\phi}|1\rangle^{b,c} \equiv L_{-1}|0,2\sqrt{2}\rangle^{x,\phi}|1\rangle^{b,c} = \tilde{Q}|0,2\sqrt{2}\rangle^{x,\phi}|0\rangle^{b,c} \tag{42}$$

is one of the spurious states at discrete values of momenta that have been studied in [17]. The complete contribution to the string field $|\varphi\rangle$ from $n = 0$ fields reads:

$$|\varphi^0\rangle = \int d\varepsilon A_0(\varepsilon)|Y_{1,0}^{\varepsilon}\rangle + \frac{1}{\sqrt{2}}D_0(2\sqrt{2})\tilde{Q}|0,2\sqrt{2}\rangle^{x,\phi}|0\rangle^{b,c}. \tag{43}$$

For $n \neq 0$ eq.(40) yields:

$$A_n(\varepsilon) = \frac{-1}{\sqrt{2}n}(\varepsilon - 2\sqrt{2})D_n(\varepsilon). \tag{44}$$

After the suitable redefinition of the field D_n one gets the contribution into $|\varphi\rangle$:

$$|\varphi^1\rangle = \int d\varepsilon \sum_{n\neq 0} D_n(\varepsilon)[\frac{2L_0}{\sqrt{2}n}\alpha_{-1}c_1 - 2\tilde{Q}]|\sqrt{2}n,\varepsilon\rangle^{x,\phi}|0\rangle^{bc} \tag{45}$$

As it must be, the decomposition of the string field being restricted on mass-shell strictly reproduces the well known results on relative cogomology [13].

The lesson that one learns from this example is the following, any string field $|\varphi\rangle$ obeying the gauge condition (31) can be presented in the form:

$$|\varphi\rangle = \int d\varepsilon [A_{J,n}(\varepsilon)|Y_{J,n}^{\varepsilon}\rangle + D_{a,n}(\varepsilon)|a,\sqrt{2}n,\varepsilon\rangle], \tag{46}$$

where a is a label of states and each state $|a,\sqrt{2}n,\varepsilon\rangle$ being restricted on-shell is $\tilde{Q}$ exact. This decomposition provides the proper integration variables in this gauge as $A(\varepsilon)$ and $D(\varepsilon)$.

In the following we shall use for the decomposition (46) the shorthand

$$|\varphi\rangle = |A\rangle + |D\rangle. \tag{47}$$

With this notation the gauge fixed quadratic action can be written as

$$S_0 = \int A * c_0 L_0 A + 2\int A * c_0 L_0 D \tag{48}$$

$$+ \int D * c_0 L_0 D + \int \psi * c_0 M \psi \tag{49}$$

The action defines three types of internal lines and it is obvious that only $\langle A, A\rangle$ and $\langle D, D\rangle$ lines can be relevant for the most singular graphs.

Let us define the projection operator:

$$P(\varepsilon)|\varphi(\varepsilon)\rangle = |D(\varepsilon)\rangle \tag{50}$$

which can be expanded over the eighenspaces of the mass level operator $\hat{N}$

$$P = \sum_{N=1}^{\infty}\sum_{n=-\infty}^{\infty} P_{N,n}(\varepsilon), \tag{51}$$

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where N is the eighenvalue of $\hat{N}$. For ε on-shell

$$2n^2 - (\varepsilon - \sqrt{2})^2 + 2N = 0, \tag{52}$$

i.e. for

$$\varepsilon = \varepsilon_{N,n}^\sigma = \sqrt{2}(1 - \sigma\sqrt{n^2 + N}), \quad \sigma = \pm, \tag{53}$$

one has due to eq.(50)

$$P_{N,n}(\varepsilon_{N,n}^\sigma) = \tilde{Q} \times R_{N,n}^\sigma = Q \cdot R_{N,n}^\sigma \tag{54}$$

for some operators $R_{N,n}^\sigma$.

Let us examine the singular structure for $\langle D, D \rangle$ lines. Writing down the projector (51) explicitly one has for the D-fields propagator:

$$P^+ \frac{b_0}{L_0} P = \sum_{N,n} P_{N,n}^+(e) \frac{2b_0}{2n^2 - (e - \sqrt{2})^2 + 2N} P_{N,n}(e). \tag{55}$$

For e in some neighborhood of a spectral point e_0 $2n_0^2 - (e_0 - \sqrt{2})^2 + 2N_0 = 0$ one has for the numerator owing to eq.(54):

$$
\begin{aligned}
P_{N_0,n_0}^+(e) b_0 P_{N_0,n_0}(e) &= [Q R_{N_0,n_0}(e_0) + O(e - e_0)]^+ b_0 [Q R_{N_0,n_0}(e_0) + O(e - e_0)] \quad (56) \\
&= R_{N_0,n_0}^+ Q b_0 Q R_{N_0,n_0} + O(e - e_0) \\
&= R_{N_0,n_0}^+ L_0 R_{N_0,n_0} + O(e - e_0).
\end{aligned}
$$

Thus, we see the cancelation of the potential pole.

Hence, we conclude that only the trees build from $\langle A, A \rangle$ lines can produce the desired singularities. The quadratic part of the action for A fields reads (see eq.(46))

$$S_0(A) = \frac{1}{2} \int d\varepsilon \sum_{J=0}^{\infty} \sum_{n=-J}^{J} A_{J,n}(-\varepsilon + 2\sqrt{2})[(\varepsilon - \sqrt{2})^2 - 2J^2)]A_{J,n}(\varepsilon) \tag{57}$$

To obtain the interaction term S_{int} recall that it is defined by the correlation function on Witten string configuration R_W:

$$_1\langle _2\langle _3\langle V|Y_{J_1,n_1,\sigma_1}^{\varepsilon_1}\rangle_1|Y_{J_2,n_2,\sigma_2}^{\varepsilon_2}\rangle_2|Y_{J_3,n_3,\sigma_3}^{\varepsilon_3}\rangle_3 = \langle Y_{J_1,n_1}^{\varepsilon_1}(w_1)Y_{J_2,n_2}^{\varepsilon_2}(w_2)Y_{J_3,n_3}^{\varepsilon_3}(w_3)\rangle_{R_W}. \tag{58}$$

After a suitable conformal transformation to upper half-plain it is calculated to give:

$$S_{int}(A) = \frac{2}{3} \int d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2\sqrt{2})$$

$$\cdot \sum_{J_1 n_1} \sum_{J_2 n_2} \sum_{J_3 n_3} F_{J_1,n_1;J_2,n_2;J_3,n_3}(\varepsilon_1, \varepsilon_2, \varepsilon_3) A_{J_1 n_1}(\varepsilon_1) A_{J_2 n_2}(\varepsilon_2) A_{J_3 n_3}(\varepsilon_3), \tag{59}$$

where

$$F_{J_1,n_1;J_2,n_2;J_3,n_3}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \lim_{x \to \infty} x^{2(\varepsilon_1 - \sqrt{2})^2 - 4J_1^2} \prod_{r=1}^{3} (\frac{4}{3\sqrt{3}})^{(-\varepsilon_2 - \sqrt{2})^2 + 2J_2^2} \langle Y_{J_1,n_1}^{\varepsilon_1}(x)Y_{J_2,n_2}^{\varepsilon_2}(1)Y_{J_3,n_3}^{\varepsilon_3}(0)\rangle. \tag{60}$$

It is not difficult to realize that the action $S = S_0 + S_{int}$ strictly reproduces the formula (30) for the most singular part by following the usual Feynman rules.

4 Conclusion

Let us make some general comments on the problem of constructing the tree-level S-matrix for 2d string discussed in this paper.

Our system has common features with the usual quantum field theory in the finite volume. Indeed, in Quantum Field Theory in the finite volume one deals with discrete states with momenta lying on a lattice. The S-matrix approach for such a theory is not adequate since the free Hamiltonian has the discrete spectrum and as it is well known from Quantum Mechanics, the S-matrix approach is not a proper one for systems with a discrete spectrum. In our case the situation is more complicated since we have to deal with the system resembling a field theory model with particle decays. These facts have to push us to consider the off-shell S-matrix approach and String Field Theory as a starting point.

In the paper we restrict ourself with consideration of tree-level scattering amplitudes. This is related with the fact that according to relation (35) the most singular parts of the off-shell S-matrix are concentrated in the non-loop diagramms.

Note also that there is an alternative possibility to avoid the appearance of singularities at all. It consists in introduction of some extra factors for vertex operators. As it has been shown above the singular part of each tree graph is the product of poles and the structure constants, which are subject of OPE of external states vertex operators. To get total amplitudes one has to make all permutations of external states. Since all of the structure constant (7) entering in OPE of vertex operators with cocycle factors are antisymmetric in low indices, it is follows that the most singular parts of the graphs will be cancelled in the total scattering amplitudes. So in the case when the external states are represented by vertex operators with cocycle factors, the decay situation is not admitted. However behind this construction we have not selfconsistent scheme such as String Field Theory.

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